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ON THE DIRECT PRODUCT IN THE THEORY OF FINITE GROUPS*

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1. THIS paper is an investigation into the formation of groups by the process known as the "direct product." The calculus introduced by Frobenius is used throughout with the addition of the sign $A \frown B$ to denote the largest sub-complex which is common to the complexes A and B ; simple isomorphism between A and B is denoted by $A \sim B$.

Groups are represented by Gothic letters and their elements by the corresponding Roman capitals; ordinary numbers are represented by small Roman letters, and in particular the order of a group is denoted by the same letter of the alphabet as the group itself. Thus the elements of a group A are denoted by $A_1, A_2, \dots, A_a$, a being its order. The identity is denoted as usual by I .

2. If a group H has two sub-groups A and B such that (1) $H = AB$, (2) $A \frown B = I$, (3) every element of A is commutative with every element of B , then H is said to be the *direct product* of A and B , and A and B are called *direct factors* of H .

The relation between H , A and B is conveniently† denoted by $H = A \times B$. Where the meaning is otherwise clear, the adjective "direct" may be omitted and A and B simply called factors of H . In this paper the term "factor" will be used solely in this sense. A group whose only factors are itself and the identity is said to be *prime*.

It may be noticed here that, instead of saying that every element of A is commutative with every element of B , it is sufficient to state that A and B are invariant.‡

3. If G is any subgroup of $H = A \times B$, every element of G can be expressed in the form $G_r = A_{a_r} B_{b_r}$. The two sets A_{a_r} and B_{b_r} ($r = 1, 2, \dots, g$) form subgroups of A and B respectively which are isomorphic with G . These

* Read before the American Mathematical Society, September 10, 1908.

† Cf. Hurwitz, *Mathematische Annalen*, vol. 45 (1894), p. 389.

‡ Cf. Dyck, *Mathematische Annalen*, vol. 22 (1883), p. 97, and Burnside, *Theory of Groups of Finite Order* (1897), p. 44.

subgroups, which are of constant occurrence, are conveniently denoted by A_g and B_g . These groups are, of course, only defined with reference to some particular method of breaking up H into factors. Thus if $H = A \times B = A \times C$, the first factorisation gives $A_b = I$, while this is not the case when the second factorisation is taken, unless $B = C$, which is by no means necessarily the case.

THEOREM 1. *If G is a non-abelian invariant sub-group of $A \times B$, either $A \frown G > I$ or $B \frown G > I$.*

If possible let $A \frown G = I = B \frown G$. Then evidently $A_g \sim G \sim B_g$. Since A and G are invariant and $A \frown G = I$, every element of A is commutative with every element of G . Hence A_g is abelian: for if $G = A_1 B$ and A_2 is any element of A , then $A_2 A_1 B = A_2 G = G A_2 = A_1 B A_2 = A_1 A_2 B$, and therefore $A_2 A_1 = A_1 A_2$. But, since $G \sim A_g$ and G is not abelian, this is impossible; hence the theorem is valid.

THEOREM 2. *If $A \times B = A \times C$, then $B \sim C$.*

For B and C are both simply isomorphic with the quotient group of $A \times B$ with respect to A . We may also notice here that $B_c = B$, since A and C are both contained in AB_c .

If $BC = B \times A'$ where $A' \leq A$, it is easily shown, as in the proof of the last theorem, that A' is a sub-group of the central of A . If then the central of A reduces to the identity, we have $B = C$.

4. If $H = A_1 \times A_2 \times \dots \times A_p$ is an expression of H as the product of prime factors, $A_1, A_2, \dots, A_p$ is said to be a *prime composition series* of H . Two prime composition series are *equivalent* if they have the same number of terms and if their terms can be so arranged that corresponding ones are simply isomorphic.

THEOREM 3. *Any two prime series of a group are equivalent; and the correspondence of the terms can be so arranged that $A_b = A$ and $B_a = B$, if A and B are corresponding terms.**

Assume that the theorem is true for groups whose order is less than the order of the given group H , and let $A_1, A_2, \dots, A_p$ and $B_1, B_2, \dots, B_q$ be two prime series for H .

* This theorem is of course well known in the case of abelian groups. In case of non-abelian groups it is implied in the known theorem that a direct product may be represented as a product of substitution groups written in distinct sets of letters. Cf. Miller, *Transactions of the Amer. Math. Society*, vol. 1 (1900), p. 71.

Let the factors be so arranged that $a_1 \leq b_1$ and consider the group

$$\begin{aligned} G &= A_1 (B_2 \times \cdots \times B_q) \\ &= B'_1 \times B_2 \cdots \times B_q, \end{aligned}$$

where $B'_1 \leq B_1$.

If $G = H$, then $B'_1 = B_1$, and, since $a_1 \leq b_1$, we must have

$$H = A_1 \times B_2 \times \cdots \times B_q.$$

Theorem 2 then gives $A_1 \sim B_1$ and $A_2 \times \cdots \times A_p \sim B_2 \times \cdots \times B_q$, whence the first part of the theorem holds for H by hypothesis. The second part of the theorem obviously holds as regards A_1 and B_1 ; it can be shown to hold for the other factors as follows.

Let $C = A_2 \times \cdots \times A_p$ and $D = B_2 \times \cdots \times B_q$. The relations given above then become

$$H = A_1 \times C = B_1 \times D = A_1 \times D$$

and $p = q$, $c = d$.

Now $AC_{b_2}C_{b_3} \cdots C_{b_q}$ contains $A_1, B_2, \dots, B_q$ and is therefore equal to H , hence $C_{b_2} \cdots C_{b_q} = C$. But $c = d = b_2b_3 \cdots b_q$ and hence $C = C_{b_2} \times C_{b_3} \times \cdots \times C_{b_q}$, thus giving a prime series for C . C_{b_r} is therefore by hypothesis simply isomorphic with say A_r and $(A_r)_{c_{b_r}} = A_r$. But $B_r < A_1C_{b_r}$ and thence $(A_r)_{c_{b_r}} = (A_r)_{b_r}$. The second part of the theorem is therefore also true for H .

Suppose now that $G < H$. It follows by hypothesis that A_1 being a prime factor of G is simply isomorphic with B_s for some value of $s > 1$ and that $(A_1)_{b_s} = A_1$, $(B_s)_{a_1} = B_s$, or else that $A_1 \sim B'_1$.

If $A_1 \sim B'_1$, we have $A_1 \cap B_2 \times \cdots \times B_q = I$, since $(B'_1)_{a_1} = B'_1$, and therefore

$$\frac{H}{A_1} = A_2 \times \cdots \times A_p = \frac{B_1}{B'_1} \times B_2 \times \cdots \times B_q.$$

Hence, by hypothesis, to any factor B_s ($s > 1$) there corresponds a factor A_s satisfying the conditions laid down in the theorem, viz.:

$$A_s \sim B_s, (A_s)_{b_s} = A_s, \text{ and } (B_s)_{a_s} = B_s.$$

Putting $s = 1$, for convenience, this gives immediately

$$A_1 \times A_2 \times \cdots \times A_p = B_1 \times A_2 \times \cdots \times A_p = B_1 \times B_2 \times \cdots \times B_q,$$

and hence as above we find that the theorem holds for H .

The theorem is obviously true when the order h of H is unity and hence, by induction, it is true for any group.

In some cases the factors of a group are unique. This is so, for instance, when their orders are relatively prime, or more generally if the orders of their centrals are relatively prime. The following theorem, along with the corollaries to theorem 2, shows the connection between the factors when they are not unique.

THEOREM 4. *If A and B are any two factors of H , A being prime, then if $A \frown B > I$, B has a factor which is simply isomorphic with A .*

Since B is a factor, $AB = C \times B$, and if $A \frown B > I$, then $c < a$. Hence C is not simply isomorphic with A , and therefore, by the previous theorem, B has a factor which is simply isomorphic with A .

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